Calculus Assignment Solutions

1. Using the limit process: $f(x) = \dfrac{1}{x^2}$

By definition,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h}.$$

Combine the fraction:

$$\frac{\frac{x^2-(x+h)^2}{x^2(x+h)^2}}{h} = \frac{-2xh - h^2}{h\,x^2(x+h)^2} = \frac{-2x - h}{x^2(x+h)^2}.$$

Let $h \to 0$:

$$f'(x) = \frac{-2x}{x^2 \cdot x^2} = -\frac{2}{x^3}.$$

2. Find the derivative: $f(x) = \dfrac{2}{\sqrt[3]{x}} + 3\cos x$

Rewrite:

$$f(x) = 2x^{-\frac{1}{3}} + 3\cos x.$$

Differentiate term by term:

$$\frac{d}{dx}\left(2x^{-\frac{1}{3}}\right) = 2 \cdot \left(-\tfrac{1}{3}\right) x^{-\frac{4}{3}} = -\tfrac{2}{3}x^{-\frac{4}{3}},$$

$$\frac{d}{dx}(3\cos x) = -3\sin x.$$

Hence,

$$f'(x) = -\frac{2}{3}x^{-\frac{4}{3}} - 3\sin x.$$

3. Higher order derivative: Given $f^{(4)}(t) = t\cos t$, find $f^{(5)}(t)$

Differentiate once:

$$f^{(5)}(t) = \frac{d}{dt}\big(t\cos t\big) = \cos t - t\sin t.$$

4. Second derivative: $f(x) = \sec^2(\pi x)$

Let $u = \pi x$. Then $f(x) = (\sec u)^2$. Using the chain rule,

$$f'(x) = 2\sec u \cdot (\sec u \tan u) \cdot u' = 2\pi\ \sec^2(\pi x)\tan(\pi x).$$

Differentiate again. Write $g(u) = \sec^2 u\ \tan u$ so that $f'(x) = 2\pi\, g(u)$ and

$$\frac{dg}{du} = (2\sec^2 u \tan u)\tan u + \sec^2 u \cdot \sec^2 u = 2\sec^2 u\ \tan^2 u + \sec^4 u.$$

Thus

$$f''(x) = 2\pi \cdot \frac{dg}{du} \cdot u' = 2\pi^2\big(2\sec^2(\pi x)\tan^2(\pi x) + \sec^4(\pi x)\big).$$

Using $\sec^2 u = 1 + \tan^2 u$,

$$f''(x) = 2\pi^2\ \sec^2(\pi x)\ \big(1 + 3\tan^2(\pi x)\big).$$